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THE EFFECTS OF MATERIAL ROTATIONS IN TENSION–TORSION TESTING

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Abstract—A new vector technique is presented for calculations appropriate to multiaxial testing configurations. An especially important feature of the technique is the explicit inclusion of material rotations associated with simple shear. The technique is applied to the prediction of axial extension rates accompanying torsional deformation . The predictions of two types of inelastic constitutive theories are compared.

It is shown that analysis of axial extension generated by torsion is a very sensitive test of the multiaxial form of inelastic constitutive equations.

INTRODUCTION

The simplest and most usual method for investigation of the inelastic behavior of materials is that of the uniaxial tension test. Of course this method subjects the test specimen only to a single type of loading. For isotropic materials the inelastic strain rate in this test is also congruent to the deviator of the loading stress. The critical investigation of consitutive relations for inelastic deformation requires testing under a broader range of loading stresses than is provided in the uniaxial configuration.

The most common method for introducing additional stress states is the testing of thin walled cylinders under combined tension and torsion. This technique is referred to as biaxial testing since the form of the stress tensor is determined by two independent loading parameters. Although the quantity of biaxial testing is still very much smaller than that done under uniaxial conditions, the number of biaxial testing programs has increased in recent times. A review of much of the significant work of this type has been published recently by Hecker[1].

There is an important aspect of the torsion deformation mode in tension–torsion testing that has been consistently ignored in almost all of such testing programs. It is that the torsion deformation is not one of pure shear strain but rather is that of simple shear which includes material rotation. This fact complicates the interpretation of tension–torsion data, and it can lead to profound errors in the identification of the parameters in inelastic constitutive equations. It is necessary, therefore, that a reasonably general procedure be available to account for the effects of material rotations if reliable conclusions are to be drawn from tension–torsion test data.

The purpose of the present paper is to develop such a procedure. It will be seen in what follows that the detailed application of the procedure to test data depends to some extent on the character of the constitutive equations themselves. Thus it is not possible to provide an automatic method of analysis that applies in the same way to all classes of constitutive equations. For this reason we shall develop the theory in parts. Some of the parts are general, but some parts of the theory can be applied only with explicit attention to the detailed form of the operative constitutive equations.

We shall be concerned here only with the inelastic strain components and with incremental flow relations. Most of the considerations have to do with real time flow rate relations, and time independent representations will appear only as limiting forms of flow rate theories.

An additional mode of testing that introduces multiaxiality of loading is that of internal pressure in the testing of thin-walled cylinders. Our general formulation will include this additional degree of freedom in the admissable loading, but the special cases considered will be restricted to the tension–torsion configuration.

In the exposition that follows we shall employ a novel mathematical representation for the variables of multiaxial testing that replaces the somewhat cumbersome tensorial description by a simpler vectorial one. Since this technique is novel, we shall first develop it in some detail. We shall then apply the method to the representation of the effects of material rotations on the appropriate vectors. Finally, we shall apply the resultant equations to the prediction of tension–torsion behavior for two types of constitutive equations.

We shall follow, in our treatment, the customary approximation that the cylindrical tube has walls sufficiently thin that the tube may be regarded as locally flat. Thus the specimen is equivalent to a thin planar sheet. We choose coordinate axes x_1 and x_2 in the plane such that the x_1-direction is axial and the x_2-direction is circumferential. The x_3-direction is then taken normal to the plane.

We assume explicitly in our treatment that the local geometric rotation rate that is derived from the anti-symmetric components of the velocity gradient is a material rotation rate with respect to any element of material anisotropy. Thus we assume that any local rotation rate affects the material in the same way as would a local rigid rotation. There is no clear proof available that this condition holds strictly. We intend the treatment to apply to polycrystalline materials, and the assumption appears to be reasonable for such materials.

THE 3-D DEVIATORIC VECTOR SPACE

For the treatment of material strain under conditions appropriate for tension–torsion testing of thin-walled cylinders it is convenient to employ a system of three linearly independent tensors that can be used as a basis set for a 3-dimensional vector space. In such a representation the deviatoric stress tensors and the non-elastic strain rate tensors can be treated as vectors. Although the tension–torsion loading system has only two independent components, the constitutive equations may require a third component and the strain rate may not necessarily be bivariate. Furthermore, with the third vector component, it is possible to include in the representation the additional loading component appropriate to internal pressure loads. Because of these circumstances we shall present the vector formulation for a deviatoric, symmetric tensor with the symmetry restrictions appropriate to our problems.

We wish to represent a symmetric, deviatoric tensor **A** of the following form:

$$\mathbf{A} \equiv \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}. \tag{1}$$

Since **A** is deviatoric we may choose any two of the diagonal components as independent. We elect to treat A_{11} and A_{22} as the independent components. Then

$$A_{33} = -(A_{11} + A_{22}). \tag{2}$$

Now we represent **A** as the sum of three terms.

$$\mathbf{A} = \frac{1}{2}(A_{11} - A_{22}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A_{12} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}(A_{11} + A_{22}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3}$$

It is easily verified that eqn (3) is the same as eqn (1) with the condition of eqn (2). We now write this as

$$\mathbf{A} = \frac{1}{2}(A_{11} - A_{22})\xi_1 + A_{12}\xi_2 + \frac{1}{2}(A_{11} + A_{22})\xi_3, \tag{4}$$

where we define

$$\xi_1 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{5}$$

The ξ's are manifestly deviatoric and linearly independent. We next define a scalar product for these vectors as the trace of the matrix product. That product is an invariant. In detail

$$\xi_i \cdot \xi_j \equiv \mathrm{Tr}\,\{\xi_i \xi_j\} \tag{6}$$

and it is easily verified that $\xi_i \cdot \xi_j = 0$ if $i \neq j$. When $i = j$, we have

$$\xi_1 \cdot \xi_1 = \xi_2 \cdot \xi_2 = 2, \tag{7}$$

$$\xi_3 \cdot \xi_3 = 6. \tag{8}$$

We form from these an orthonormal set $\hat{\xi}_i$ as follows:

$$\hat{\xi}_1 \equiv \frac{1}{\sqrt{2}}\xi_1, \quad \hat{\xi}_2 \equiv \frac{1}{\sqrt{2}}\xi_2, \quad \hat{\xi}_3 \equiv \frac{1}{\sqrt{6}}\xi_3, \tag{9}$$

and for these unit vectors,

$$\hat{\xi}_i \cdot \hat{\xi}_j = \delta_{ij}. \tag{10}$$

Now for any tensor **A** we define its vector representation $\underset{\sim}{A}$ as

$$\underset{\sim}{A} \equiv A_1\hat{\xi}_1 + A_2\hat{\xi}_2 + A_3\hat{\xi}_3, \tag{11}$$

and

$$A_1 \equiv \frac{1}{\sqrt{2}}(A_{11} - A_{22}), \tag{12}$$

$$A_2 \equiv \sqrt{(2)}A_{12}, \tag{13}$$

$$A_3 \equiv \sqrt{\left(\frac{3}{2}\right)}\,(A_{11} + A_{22}). \tag{14}$$

In terms of the components A_i, we may readily express the second invariant A of **A** defined as

$$A \equiv +\sqrt{(\mathrm{Tr}\,\{\mathbf{AA}\})}. \tag{15}$$

In terms of the scalar product this becomes

$$A = +\sqrt{(\underset{\sim}{A} \cdot \underset{\sim}{A})} \tag{16}$$

$$= +\sqrt{(A_1^2 + A_2^2 + A_3^2)}. \tag{17}$$

Thus A is the magnitude of the vector $\underset{\sim}{A}$.

Material rotations and the $\hat{\xi}$-space

We shall be concerned with material tensors **A** that will share the rotations that material elements may undergo. It is sufficient to restrict our considerations to rotations in the x_1, x_2-plane about the x_3-axis. We shall hold the basis set $\{\hat{\xi}_i\}$ fixed and seek the effect of space rotations ω on the components A_i. First, since the spatial x_3-axis is undisturbed, it is evident that A_3 will remain unchanged. The rotations then concern only A_1 and A_2. As is illustrated in the Fig. 1, the small material rotation $\mathrm{d}\omega$ induces the small vector rotation $2\mathrm{d}\omega$ in the $\hat{\xi}_1, \hat{\xi}_2$-plane. For small rotations this gives the component transformations

$$\frac{\mathrm{d}}{\mathrm{d}\omega}A_1 = -2A_2, \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}\omega}A_2 = +2A_1, \tag{19}$$

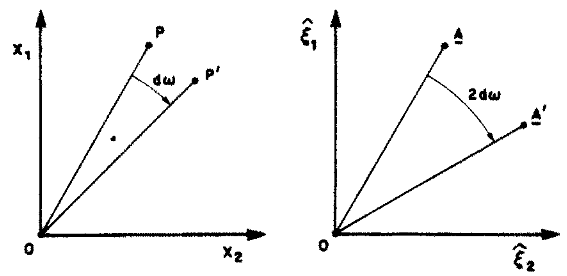

Fig. 1. Representation of the rotation of a material vector $\underset{\sim}{A}$ in the $\hat{\xi}_1$, $\hat{\xi}_2$-plane that is induced by the spatial material rotation $d\omega$ in the x_1, x_2-plane.

and

$$\frac{d}{d\omega} A_3 = 0. \tag{20}$$

Then for the vector $\underset{\sim}{A}$,

$$\frac{d}{d\omega} \underset{\sim}{A} = -2A_2\hat{\xi}_1 + 2A_1\hat{\xi}_2. \tag{21}$$

For the case of torsion of thin walled cylinders, the material deformation is that of simple shear which involves both symmetric shear and rotation. The x_2-coordinate is taken to be the circumferential one and the material velocities v_1 and v_2 are independent of x_2. Then if $\dot{\gamma}$ is the simple shearing rate

$$v_2 = \dot{\gamma}x_1 \tag{22}$$

$$v_{2,1} = \dot{\gamma} \tag{23}$$

$$\dot{\epsilon}_{12} = \frac{1}{2}(v_{2,1} + v_{1,2}) \tag{24}$$

$$= \frac{1}{2}\dot{\gamma} \tag{25}$$

$$\dot{\omega} = \frac{1}{2}(v_{2,1} - v_{1,2}) \tag{26}$$

$$= \frac{1}{2}\dot{\gamma} = \dot{\epsilon}_{12}. \tag{27}$$

The positive sense of $\dot{\omega}$ is rotation of x_1 into x_2.

Now any material tensor **A** that has a time rate of change $(d\mathbf{A}/dt)_{mat}$ in the material reference frame will have a time rate of change $\dot{\mathbf{A}}$ in the fixed reference frame (laboratory frame) given by

$$\dot{\mathbf{A}} = (d\mathbf{A}/dt)_{mat} + \dot{\omega}\, d\mathbf{A}/d\omega. \tag{28}$$

APPLICATION TO CONSTITUTIVE MODELS

We wish to find the predictions of two classes of constitutive equations for the resultant deformation under multiaxial test conditions that include torsion. The effect of the rotations will appear in these models from the material anisotropy induced by the deformation. This anisotropy is represented in both models by an internal stress (or stored strain) which gives an orientation to the material elements. Since that internal variable is tensorial, its laboratory frame time rate of change will follow eqn (28). The resultant effect however is somewhat different for the two constitutive models and so we must treat each one separately. We shall first carry out the calculation for the internal state variable theory proposed by Hart[2]. This theory has already been applied to this problem by VanArsdale *et al.*[3]. We shall restate that problem in the vector representation developed above and in a more general context than that of Ref. [3]. We shall then apply the same method to the kinematic hardening theory.

HART'S CONSTITUTIVE EQUATIONS

These constitutive equations are relations among the applied stress deviator $\boldsymbol{\sigma}$, the observable non-elastic strain rate $\dot{\boldsymbol{\epsilon}}$, a tensorial internal state variable $\mathbf{a}$, and a scalar state variable σ^*. In describing the theory it is convenient to employ three additional auxiliary variables $\dot{\boldsymbol{\alpha}}$, $\boldsymbol{\sigma}_a$, and $\boldsymbol{\sigma}_f$. The relationships among these are shown in the rheological diagram in Fig. 2. The intent of the diagram is to show all relationships and constraints as tensorial.

The variable $\dot{\boldsymbol{\alpha}}$ is a strain rate that represents the fully unrecoverable plastic strain rate. Its value depends on the auxiliary "internal" stress $\boldsymbol{\sigma}_a$ and on the current value of the "hardness" σ^* through relations given below. The total non-elastic strain rate $\dot{\boldsymbol{\epsilon}}$ depends immediately on the auxiliary "effective" stress $\boldsymbol{\sigma}_f$ through a non-Newtonian viscous relation. For more detailed discussion of the relations see Ref. [2]. We summarize the relations below.

The stress variables are related by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_f. \tag{29}$$

The variable $\mathbf{a}$ represents an internal stored anelastic strain and the strain rate constraint in the material frame is given by

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\alpha}} + (\mathrm{d}\mathbf{a}/\mathrm{d}t)_{\mathrm{mat}}. \tag{30}$$

since $\mathbf{a}$ is a material tensor its fixed frame time variation follows eqn (28) and we have

$$\dot{\mathbf{a}} = \dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\alpha}} + \dot{\omega}(\mathrm{d}\mathbf{a}/\mathrm{d}\omega). \tag{31}$$

We employ the notation of eqn (15) for the tensor invariant and we prescribe, for isotropic materials, that each mechanism shown in Fig. 2 is separately isotropic in terms of its immediate variables. Thus

$$\dot{\boldsymbol{\epsilon}} = (\dot{\epsilon}/\sigma_d)\boldsymbol{\sigma}_f, \tag{32}$$

$$\dot{\boldsymbol{\alpha}} = (\dot{\alpha}/\sigma_a)\boldsymbol{\sigma}_a, \tag{33}$$

$$\boldsymbol{\sigma}_a = \mathcal{M}\mathbf{a}. \tag{34}$$

The remaining equations are in terms of the scalar invariants and several functions.

$$\dot{\epsilon} = \dot{a}^*(\sigma_f/\mathcal{M})^M, \tag{35}$$

$$\ln(\sigma^*/\sigma_a) = (\dot{\epsilon}^*/\dot{\alpha})^\lambda, \tag{36}$$

$$\dot{\epsilon}^* = (\sigma^*/G)^m \cdot f \cdot \exp[-Q/RT], \tag{37}$$

$$\mathrm{d}\ln\sigma^*/\mathrm{d}\alpha = \Gamma(\sigma^*, \sigma_a). \tag{38}$$

In these equations, G is the modulus of rigidity, Q is an activation energy, R is the gas constant, T is the absolute temperature, f is an experimental frequency constant, $\dot{a}^*$ is a function of T, Γ is an experimental function of its arguments, and $\mathcal{M}$, M, m, and λ are constants.

Although the relations seem complex, they are in fact rather simple, and in our application below we shall use an even simpler low temperature limiting form which we have termed the visco-plastic limit [2].

The important point for our application to the tension–torsion problem is that all the tensor relations above may be transcribed immediately into the vector representation we have discussed. We now proceed directly to the formulation of our problem.

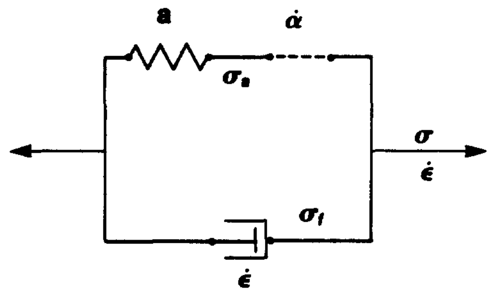

Fig. 2. A schematic diagram representing the kinematical and mechanical relations among the three elements of Hart's constitutive equations for inelastic deformation. The conventions of the diagram are those customary in rheological diagrams and tensor quantities are implied throughout. (After Hart, Ref. [1].)

TENSION-TORSION TESTING OF THIN-WALLED CYLINDERS

The testing of thin-walled cylindrical specimens under combined axial and torsional loading is substantially equivalent to the superposition of axial loading and transverse shear loading of plane specimens, and this testing configuration is conveniently treated this way. We propose to investigate the problem of an imposed shearing rate with a superposed axial stress. We are interested in the attendant axial elongation rate. The axial strain rate is not zero even when the axial stress is zero[3]. This effect, which can be thought of as second order, is a rather sensitive test of the multiaxial character of the constitutive relations for the material.

We proceed directly to the case of materials satisfying Hart's[2] constitutive equations. The equation of importance is eqn (31) which we repeat here in vectorial form,

$$\dot{\underset{\sim}{a}} = \dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\alpha}} + \dot{\omega}(\mathrm{d}\underset{\sim}{a}/\mathrm{d}\omega). \tag{31}$$

In components this becomes

$$\left.\begin{aligned} \dot{a}_1 + 2\dot{\omega}a_2 &= \dot{\epsilon}_1 - \dot{\alpha}_1, \\ \dot{a}_2 - 2\dot{\omega}a_1 &= \dot{\epsilon}_2 - \dot{\alpha}_2, \\ \dot{a}_3 &= \dot{\epsilon}_3 - \dot{\alpha}_3. \end{aligned}\right\} \tag{39}$$

Since $\dot{\omega}$ is equal to the symmetric shear rate $\dot{\epsilon}_{12}$ we have

$$\dot{\omega} = \dot{\epsilon}_{12} = \frac{1}{\sqrt{2}}\dot{\epsilon}_2, \tag{40}$$

and so

$$\left.\begin{aligned} \dot{a}_1 + \sqrt{(2)}\dot{\epsilon}_2 a_2 &= \dot{\epsilon}_1 - \dot{\alpha}_1, \\ \dot{a}_2 - \sqrt{(2)}\dot{\epsilon}_2 a_1 &= \dot{\epsilon}_2 - \dot{\alpha}_2, \\ \dot{a}_3 &= \dot{\epsilon}_3 - \dot{\alpha}_3. \end{aligned}\right\} \tag{41}$$

Our next problem is to find a suitable form for $\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\alpha}}$ for use in eqn (41). We choose to use only minimal information from the constitutive equations so that the analysis of experiment can be as model independent as possible. The first condition we use is the "normality" condition of eqn (33), which yields

$$\begin{aligned} \dot{\underset{\sim}{\alpha}} &= (\dot{\alpha}/\sigma_a)\underset{\sim}{\sigma}_a, \\ &= (\dot{\alpha}/\sigma_a)\mathcal{M}\underset{\sim}{a}, \\ &= (\dot{\alpha}/a)\underset{\sim}{a}. \end{aligned} \tag{42}$$

This leads to the equation

$$\left.\begin{aligned} \dot{a}_1 + \sqrt{(2)}\dot{\epsilon}_2 a_2 &= \dot{\epsilon}_1 - (\dot{\alpha}/a)a_1, \\ \dot{a}_2 - \sqrt{(2)}\dot{\epsilon}_2 a_1 &= \dot{\epsilon}_2 - (\dot{\alpha}/a)a_2, \\ \dot{a}_3 &= \dot{\epsilon}_3 - (\dot{\alpha}/a)a_3. \end{aligned}\right\} \tag{43}$$

We may now eliminate $\dot{\alpha}$ from eqn (43) by algebraic reduction to the set

$$\left.\begin{aligned} a_2\dot{a}_1 - a_1\dot{a}_2 + \sqrt{(2)}\dot{\epsilon}_2(a_1{}^2 + a_2{}^2) &= a_2\dot{\epsilon}_1 - a_1\dot{\epsilon}_2, \\ a_3\dot{a}_1 - a_1\dot{a}_3 + \sqrt{(2)}\dot{\epsilon}_2 a_2 a_3 &= a_3\dot{\epsilon}_1 - a_1\dot{\epsilon}_3, \\ a_2\dot{a}_3 - a_3\dot{a}_2 + \sqrt{(2)}\dot{\epsilon}_2 a_1 a_3 &= a_2\dot{\epsilon}_3 - a_3\dot{\epsilon}_2. \end{aligned}\right\} \tag{44}$$

It is easy to establish the identity

$$a_i\dot{\epsilon}_j - a_j\dot{\epsilon}_i = (1/\mathcal{M})(\sigma_i\dot{\epsilon}_j - \sigma_j\dot{\epsilon}_i), \tag{45}$$

from eqns (29), (32), and (34). Then

$$\left.\begin{aligned} a_1\dot{a}_2 - a_2\dot{a}_1 - \sqrt{(2)}\dot{\epsilon}_2(a_1{}^2 + a_2{}^2) &= \frac{1}{\mathcal{M}}(\sigma_1\dot{\epsilon}_2 - \sigma_2\dot{\epsilon}_1), \\ a_3\dot{a}_1 - a_1\dot{a}_3 + \sqrt{(2)}\dot{\epsilon}_2 a_2 a_3 &= \frac{1}{\mathcal{M}}(\sigma_3\dot{\epsilon}_1 - \sigma_1\dot{\epsilon}_3), \\ a_2\dot{a}_3 - a_3\dot{a}_2 + \sqrt{(2)}\dot{\epsilon}_2 a_1 a_3 &= \frac{1}{\mathcal{M}}(\sigma_2\dot{\epsilon}_3 - \sigma_3\dot{\epsilon}_2). \end{aligned}\right\} \tag{46}$$

We must now make further use of the constitutive equations, particularly eqns (29), (32), and (34). These yield the relation

$$\mathcal{M}\underset{\sim}{a} = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}_f \tag{47}$$

$$= \underset{\sim}{\sigma} - (\sigma_f/\dot{\epsilon})\dot{\underset{\sim}{\epsilon}}. \tag{48}$$

Since σ_f is determined entirely as a function of $\dot{\epsilon}$, we have represented $\underset{\sim}{a}$ in terms of measured variables $\underset{\sim}{\sigma}$ and $\dot{\underset{\sim}{\epsilon}}$. At this point we could convert eqn (46) to a set of equations among the components of $\underset{\sim}{\sigma}$, $\dot{\underset{\sim}{\epsilon}}$, and their time derivatives. Those equations would be rather cumbersome. Since in this paper we are concerned mainly with the problem of axial extension rates accompanying torsional deformation, and since we shall restrict our considerations to the case where $\sigma_{11} \ll \sigma_{12}$, we can introduce a useful simplification in eqn (48).

The strain rate magnitude $\dot{\epsilon}$ is

$$\dot{\epsilon} = \sqrt{(\dot{\epsilon}_1{}^2 + \dot{\epsilon}_2{}^2 + \dot{\epsilon}_3{}^2)}, \tag{49}$$

and, if $\sigma_{11} \ll \sigma_{12}$, $\dot{\epsilon}_1$ and $\dot{\epsilon}_3$ will always be much smaller than $\dot{\epsilon}_2$. Then to a very good approximation

$$\dot{\epsilon} = \dot{\epsilon}_2. \tag{50}$$

Now $\underset{\sim}{\sigma}_f$ is given by the relations

$$\left.\begin{aligned} \sigma_{f2} &= \sigma_f, \\ \sigma_{f1} &= \sigma_f(\dot{\epsilon}_1/\dot{\epsilon}_2), \\ \sigma_{f3} &= \sigma_f(\dot{\epsilon}_3/\dot{\epsilon}_2), \end{aligned}\right\} \tag{51}$$

and, since we shall consider tests for which the imposed shearing rate $\dot\epsilon_{12}$ is constant, it is convenient to use the quantity $\dot\epsilon_2\,dt$ as a path variable. We define, therefore,

$$\epsilon_1' \equiv d\epsilon_1/d\epsilon_2 = \dot\epsilon_1/\dot\epsilon_2,$$
$$\epsilon_3' \equiv d\epsilon_3/d\epsilon_2 = \dot\epsilon_3/\dot\epsilon_2,$$

and the similar relations

$$a_i' \equiv da_i/d\epsilon_2 = \dot a_i/\dot\epsilon_2,$$
$$\sigma_i' \equiv d\sigma_i/d\epsilon_2 = \dot\sigma_i/\dot\epsilon_2.$$

Now the simplified governing equations are

$$\left.\begin{aligned} \sigma_{a1} &= \mathcal{M}a_1 = \sigma_1 - \sigma_f\epsilon_1', \\ \sigma_{a2} &= \mathcal{M}a_2 = \sigma_2 - \sigma_f, \\ \sigma_{a3} &= \mathcal{M}a_3 = \sigma_3 - \sigma_f\epsilon_3', \end{aligned}\right\} \tag{52}$$

and two independent equations from eqn (46)

$$\left.\begin{aligned} a_3a_1' - a_1a_3' + \sqrt{(2)}a_2a_3 &= \frac{1}{\mathcal{M}}(\sigma_3\epsilon_1' - \sigma_1\epsilon_3'), \\ a_2a_3' - a_3a_2' + \sqrt{(2)}a_1a_3 &= \frac{1}{\mathcal{M}}(\sigma_2\epsilon_3' - \sigma_3). \end{aligned}\right\} \tag{53}$$

The quantity σ_f that appears in eqn (52) depends only on $\dot\epsilon_2$ and can be treated as a constant to be determined experimentally in any experiment for which $\dot\epsilon_2$ is constant.

TORSION WITH FIXED AXIAL LOAD

A particular case of experimental interest is that for which a thin-walled cylinder is subjected to a constant torsional deformation rate under a fixed axial load that can be zero. This corresponds to a constant imposed value for $\dot\epsilon_2$, σ_1 and σ_3. During the testing a continuous record of the shear stress τ_{12} and of $\dot\epsilon_{11}$ is maintained as a function of ϵ_{12}.

We seek a solution for $\dot\epsilon_{11}$ as it depends on $\dot\epsilon_2$, σ_1, σ_3, and the measured τ_{12}. The solution also depends on the constitutive parameters $\mathcal{M}$ and σ_f. The parameter $\mathcal{M}$ will be a constant, but the parameter σ_f will take a value that depends on $\dot\epsilon_2$. We restrict our solutions to the non-transient regime and so a_1' and a_3' can be neglected. Furthermore, since ϵ_1' and ϵ_3' are very small compared to unity we shall neglect terms involving $\epsilon_1'^2$ and $\epsilon_3'^2$.

These approximations lead to the following equations from eqns (52) and (53).

$$\sqrt{(2)}a_2a_3 = \frac{1}{\mathcal{M}}(\sigma_3\epsilon_1' - \sigma_1\epsilon_3'), \tag{54}$$

$$-a_3a_2' + \sqrt{(2)}a_1a_3 = \frac{1}{\mathcal{M}}(\sigma_2\epsilon_3' - \sigma_3), \tag{55}$$

$$\mathcal{M}a_2' = \sigma_2'. \tag{56}$$

Since the loading is a combination of shear, τ_{12}, and axial stress, τ_{11}, the components of the deviator $\boldsymbol{\sigma}$ yield the relations

$$\sigma_{11} - \sigma_{22} = \tau_{11}, \tag{57}$$

$$\sigma_{11} + \sigma_{22} = \frac{1}{3}\tau_{11}, \tag{58}$$

$$\sigma_{12} = \tau_{12}. \tag{59}$$

The vector components for $\underset{\sim}{\sigma}$ and $\underset{\sim}{\dot\epsilon}$ relate to the tensor components as follows:

$$\sigma_1 = \frac{1}{\sqrt{2}}\tau_{11}, \tag{60}$$

$$\sigma_2 = \sqrt{(2)}\tau_{12}, \qquad \sigma_f \equiv \sqrt{(2)}\tau_f, \tag{61}$$

$$\sigma_3 = \frac{1}{\sqrt{6}}\tau_{11}, \tag{62}$$

$$\dot\epsilon_1 = \frac{1}{\sqrt{2}}(\dot\epsilon_{11} - \dot\epsilon_{22}), \tag{63}$$

$$\dot\epsilon_2 = \sqrt{(2)}\dot\epsilon_{12}, \tag{64}$$

$$\dot\epsilon_3 = \sqrt{\left(\frac{3}{2}\right)}(\dot\epsilon_{11} + \dot\epsilon_{22}). \tag{65}$$

Since it is $\dot\epsilon_{11}$ that is measured, we seek the quantity

$$\dot\epsilon_{11}/\dot\epsilon_{12} = \epsilon_1' + \frac{1}{\sqrt{3}}\epsilon_3'. \tag{66}$$

Algebraic reduction of eqns (54)–(56) with the approximations noted above leads to the result

$$\dot\epsilon_{11}/\dot\epsilon_{12} = \sqrt{2}\frac{\sigma_2 - \sigma_f}{\mathcal{M}}\frac{\sigma_2 - \sigma_f}{\sigma_2} + \frac{4}{3}\frac{\sigma_1}{\sigma_2}\left[1 - \frac{\sigma_2 - \sigma_f}{\sigma_2}\frac{\sigma_2'}{\mathcal{M}}\right], \tag{67}$$

$$= 2\frac{\tau_{12} - \tau_f}{\mathcal{M}}\frac{\tau_{12} - \tau_f}{\tau_{12}} + \frac{2}{3}\frac{\tau_{11}}{\tau_{12}}\left[1 - \frac{\tau_{12} - \tau_f}{\tau_{12}}\frac{1}{\mathcal{M}}\frac{d\tau_{12}}{d\epsilon_{12}}\right]. \tag{68}$$

This formula can be made a bit simpler by using the customary notation for tension–torsion testing. We make the following transcription with the caveat that σ and $\dot\epsilon$ below are not to be comfused with the invariants defined earlier:

$$\sigma \equiv \tau_{11}, \tag{69}$$

$$\tau \equiv \tau_{12}, \tag{70}$$

$$\dot\epsilon \equiv \dot\epsilon_{11}, \tag{71}$$

$$\dot\gamma \equiv 2\dot\epsilon_{12}. \tag{72}$$

The result is now

$$\frac{d\epsilon}{d\gamma} = \frac{\dot\epsilon}{\dot\gamma} = \frac{\tau - \tau_f}{\mathcal{M}}\frac{\tau - \tau_f}{\tau} + \frac{1}{3}\frac{\sigma}{\tau}\left[1 - 2\frac{\tau - \tau_f}{\mathcal{M}}\frac{1}{\tau}\frac{d\tau}{d\gamma}\right]. \tag{73}$$

The prominent feature of this result is that it depends on a constant, $\mathcal{M}$, and on τ_f, that depends only on $\dot\gamma$.

KINEMATIC HARDENING

It is instructive to compare the predictions above for Hart's model with those of the kinematic hardening theory. Although we consider here only the simple time independent kinematic hardening model, we note that several more sophisticated recent constitutive theories deal with stored internal stress in a manner analogous to the kinematic hardening theory. Among those are the models of Miller [4], Robinson [5] and Krieg *et al.* [6].

The kinematic hardening model is given here in terms of "strain rates". The rates are however simply determined by externally imposed deformation rates or loading rates as is customary in time independent plasticity. The formal relations for this model are shown schematically in Fig. 3 and are detailed immediately below in some analogy to Hart's model treated above.

In response to deformation the material develops an internal back stress $\underset{\sim}{b}$ such that the effective stress $\underset{\sim}{\Sigma}$ that drives the plastic deformation rate $\underset{\sim}{\dot{\alpha}}$ is given by

$$\underset{\sim}{\Sigma} = \underset{\sim}{\sigma} - \underset{\sim}{b}, \tag{74}$$

where $\underset{\sim}{\sigma}$ is the applied stress as before.

The *yield condition* is, for

$$\Sigma \equiv +\sqrt{(\underset{\sim}{\Sigma} \cdot \underset{\sim}{\Sigma})}, \tag{75}$$

that

$$\left.\begin{aligned} \dot{\alpha} &= 0 \quad \text{for} \quad \Sigma < \sigma^*, \\ \dot{\alpha} &\neq 0 \quad \text{for} \quad \Sigma = \sigma^*, \end{aligned}\right\} \tag{76}$$

and the *normality condition* is given by the relation

$$\underset{\sim}{\dot{\alpha}} = (\dot{\alpha}/\Sigma)\underset{\sim}{\Sigma}, \tag{77}$$

where $\Sigma = \sigma^*$. A further requirement is the *kinematic relation*

$$\underset{\sim}{\dot{\alpha}} = \underset{\sim}{\dot{\epsilon}}, \tag{78}$$

and the internal stress $\underset{\sim}{b}$ satisfies the incremental hardening law

$$(\mathrm{d}\underset{\sim}{b}/\mathrm{d}t)_{\mathrm{mat}} = \underset{\sim}{\dot{\epsilon}} \cdot H. \tag{79}$$

The hardening parameter H may be functionally dependent on $\underset{\sim}{b}$. The fixed frame time rate of change of $\underset{\sim}{b}$ is then

$$\underset{\sim}{\dot{b}} = H\underset{\sim}{\dot{\epsilon}} + \dot{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega}\underset{\sim}{b}. \tag{80}$$

We now detail eqn (80) as follows:

$$\left.\begin{aligned} \dot{b}_1 + \sqrt{(2)}\dot{\epsilon}_2 b_2 &= H\dot{\epsilon}_1, \\ \dot{b}_2 - \sqrt{(2)}\dot{\epsilon}_2 b_1 &= H\dot{\epsilon}_2, \\ \dot{b}_3 &= H\dot{\epsilon}_3. \end{aligned}\right\} \tag{81}$$

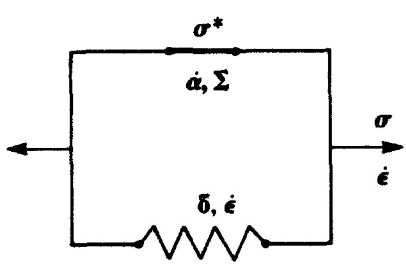

Fig. 3. A schematic diagram representing the kinematical and mechanical relations among the deformation elements for the kinematic hardening constitutive model.

We seek a "steady state" solution for which $\dot{b}_1 = \dot{b}_3 = 0$ under the conditions of an imposed $\dot{\epsilon}_2$, σ_1, and σ_3. We divide the equation by $\dot{\epsilon}_2$ as before and employ the same prime notation. Then

$$\left.\begin{aligned} \sqrt{(2)}b_2 &= H\epsilon_1', \\ b_2' - \sqrt{(2)}b_1 &= H, \\ 0 &= H\epsilon_3'. \end{aligned}\right\} \tag{82}$$

From eqns (77) and (78) we obtain

$$\epsilon_1' = (\sigma_1 - b_1)/(\sigma_2 - b_2); \qquad \epsilon_3' = (\sigma_3 - b_3)/(\sigma_2 - b_2), \tag{83}$$

and so, from eqn (82) the condition $\epsilon_3' = 0$ yields

$$(\sigma_3 - b_3) = 0. \tag{84}$$

The yield condition is

$$\begin{aligned} \sigma^* &= \sqrt{[(\sigma_1 - b_1)^2 + (\sigma_2 - b_2)^2 + (\sigma_3 - b_3)^2]} \\ &= (\sigma_2 - b_2)\sqrt{\left[1 + \left(\frac{\sigma_1 - b_1}{\sigma_2 - b_2}\right)^2\right]} \\ &= (\sigma_2 - b_2)\sqrt{(1 + \epsilon_1'^2)}. \end{aligned} \tag{85}$$

Then, from eqn (85),

$$b_2 = \sigma_2 - \frac{\sigma^*}{\sqrt{(1 + \epsilon_1'^2)}}, \tag{86}$$

and the first line of eqn (82) becomes

$$\epsilon_1' = \sqrt{(2)}\sigma_2/H - \sqrt{(2)}\sigma^*/(H\sqrt{(1 + \epsilon_1'^2)}), \tag{87}$$

We expand the square root to second degree in the small quantity ϵ_1'.

$$\epsilon_1' = \sqrt{(2)}(\sigma_2 - \sigma^*)/H + \frac{1}{2}\sqrt{(2)}(\sigma^*/H)\epsilon_1'^2. \tag{88}$$

By the same reduction to observed variables as before, this becomes

$$\mathrm{d}\epsilon/\mathrm{d}\gamma = (\tau - \tau^*)/H + 2(\tau^*/H)(\mathrm{d}\epsilon/\mathrm{d}\gamma)^2, \tag{89}$$

and finally, to a good approximation,

$$\mathrm{d}\epsilon/\mathrm{d}\gamma = \frac{\tau - \tau^*}{H}\left[1 + 2\frac{\tau^*}{H}\frac{\tau - \tau^*}{H}\right]. \tag{90}$$

In this result, τ^* is the initial yield stress in shear and H is the direct shear strain hardening given by

$$H = 2(\mathrm{d}\tau/\mathrm{d}\gamma). \tag{91}$$

DISCUSSION

The result of this computation of the axial extension accompanying torsional deformation of thin walled cylinders shows the experimental configuration to be a sensitive test of the three dimensional form of the inelastic constitutive equations. The principal results for low homologous temperature testing are contained in eqn (73) for Hart's constitutive equations and eqn (90) for constitutive equations of the kinematic hardening type. For immediate reference we repeat these.

$$d\epsilon/d\gamma = \frac{\tau-\tau_f}{\mathcal{M}}\frac{\tau-\tau_f}{\tau} + \frac{1}{3}\frac{\sigma}{\tau}\left[1 - 2\frac{\tau-\tau_f}{\mathcal{M}}\frac{1}{\tau}\frac{d\tau}{d\gamma}\right], \tag{73}$$

$$d\epsilon/d\gamma = \frac{\tau-\tau^*}{H}\left[1 + 2\frac{\tau^*}{H}\frac{\tau-\tau^*}{H}\right]. \tag{90}$$

The most obvious difference is the lack of explicit dependence on the axial load σ in the result for kinematic hardening. Actually there is some dependence on σ for that case through the dependence of τ on σ. In eqn (73) $\mathcal{M}$ is explicitly required to be a constant that is an anelastic modulus of the same order as the shear modulus G. In eqn (90) H need not be a constant (non-linearity is possible, although most published theories would have H constant), however it must be close at each stage of deformation to the condition

$$H = 2(d\tau/d\gamma),$$

since it is the kinematic hardening rate parameter. It is clear, nevertheless, that there might be some ambiguity in fitting the kinematic hardening prediction to measured results unless some sharp specification can be made for the rule whereby H is validated. We leave that question for another paper that deals with the analysis of some explicit experiments. We note, nevertheless, that at high hardening levels H must vary more slowly and so may be more nearly constant for significant ranges of γ. It is to be hoped then that in such ranges it may be possible to discriminate clearly between the two types of theory.

Note specially that the problem has been solved in each case above in terms of the control parameters $\dot{\gamma}$ and σ *and of the measured variable* τ. This procedure makes the resultant prediction of $\dot{\epsilon}$ less dependent on the detailed form of the constitutive equations. In the case of Hart's constitutive equations, this makes the test capable of determining $\mathcal{M}$ and τ_f with high precision.

CONCLUSIONS

A useful new technique for dealing with the tensor variables in multiaxial testing has been presented. The technique replaces the tensor equations with relations in a three-dimensional vector space. The effect of material rotations associated with torsional deformation is explicitly included.

The axial extension accompanying torsional deformation has been computed for two classes of inelastic constitutive equations in a form that permits easy comparison with experiment. It is proposed that a clear distinction can be made between the two types of constitutive theory by analysis of this phenomenon.

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